

EFFECTIVE FAITHFUL TROPICALIZATIONS ASSOCIATED TO ADJOINT LINEAR SYSTEMS

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ABSTRACT. Let R be a complete discrete valuation ring of equi-characteristic zero with fractional field K . Let X be a connected, smooth projective variety of dimension d over K , and let L be an ample line bundle over X . We assume that there exist a regular strictly semistable model \mathcal{X} of X over R and a relatively ample line bundle \mathcal{L} over \mathcal{X} with $\mathcal{L}|_X \cong L$. Let $S(\mathcal{X})$ be the skeleton associated to \mathcal{X} in the Berkovich analytification X^{an} of X . In this article, we study when $S(\mathcal{X})$ is faithfully tropicalized into tropical projective space by the adjoint linear system $|L^{\otimes m} \otimes \omega_X|$. Roughly speaking, our results show that, if m is an integer such that the adjoint bundle is basepoint free, then the adjoint linear system admits a faithful tropicalization of $S(\mathcal{X})$.

1. INTRODUCTION

Let X be a connected, smooth projective variety defined over a field K , let L be an ample line bundle over X , and let ω_X be the canonical line bundle over X . If X is one dimensional and $\deg(L) \geq 2g(X) + 1$, where $g(X)$ is the genus of X , then global sections of L give an embedding of X into projective space. In higher dimension, adjoint bundles $L^{\otimes m} \otimes \omega_X$ can be viewed as the generalization of line bundles of large degree on curves, and there have been many important studies on when global sections of $L^{\otimes m} \otimes \omega_X$ give an embedding (or injection) of X into projective space (see, for example, [24, II, 10.4]).

We consider an analogue for Berkovich skeleta in non-Archimedean geometry, and we study when global sections of line bundles give a faithful tropicalization of a Berkovich skeleton into tropical projective space. In [19], we have studied a one-dimensional case. In this paper, with an assumption on the existence of a model, we consider a faithful tropicalization by global sections of adjoint bundles in arbitrary dimension.

In the following, we assume that K is a complete discrete valuation field. We denote by R the ring of integers and by k the residue field. We assume that R has equi-characteristic zero and that X admits regular strictly semistable model \mathcal{X} of X over R . (If we replace R by a suitable finite extension, then there always exists such \mathcal{X} , see §2.1 for details.) Let X^{an} denotes the analytification of X in the sense of Berkovich. Let $S(\mathcal{X}) \subset X^{\text{an}}$ be the Berkovich skeleton associated to \mathcal{X} , which carries a piecewise integral rational-affine structure and is homeomorphic to the dual intersection complex of the special fiber \mathcal{X}_s (see §2.3 for details).

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We say that $S(\mathcal{X})$ has a *faithful tropicalization associated to the linear system $|L|$* if there exist nonzero global sections $s_0, \dots, s_n \in H^0(X, L)$ with the following properties:

$$(1.1) \quad \varphi : X^{\text{an}} \longrightarrow \mathbb{TP}^n, \quad p = (p, |\cdot|) \mapsto (-\log |s_0(p)| : \dots : -\log |s_n(p)|)$$

is a morphism into tropical projective space \mathbb{TP}^n and the restriction of φ to $S(\mathcal{X})$ is a homeomorphism onto its image preserving the piecewise integral rational-affine structures. (For the tropical projective space and faithful tropicalizations, see §2.4 and §2.5 for details.)

For a positive integer d , we set

$$(1.2) \quad \phi(d) := \min \left\{ m_0 \in \mathbb{Z} \left| \begin{array}{l} \text{For any smooth projective variety } Z \text{ with } \dim Z \leq d \text{ over} \\ \text{a field of characteristic zero, and for any ample line bundle} \\ N \text{ over } Z, N^{\otimes m} \otimes \omega_Z \text{ is basepoint free for any } m \geq m_0. \end{array} \right. \right\},$$

where ω_Z denotes the canonical line bundle of Z . By Angehrn and Siu [1], one has $\phi(d) \leq d(d+1)/2 + 1$, and by a subsequent work of Heier [16], one has $\phi(d) \leq O(d^{4/3})$. Fujita's conjecture concerns the optimal $\phi(d)$, which asks if $\phi(d) = d+1$ holds true, and is known to be true in dimension $d = 2, 3, 4$ (see Reider [31], Ein–Lazarsfeld [11], and Kawamata [20]).

Let us state our main theorem.

Theorem 1.1. *Let R be a complete discrete valuation ring of equi-characteristic zero with fractional field K . Let X be a connected, smooth projective variety of dimension d over K , and let L be an ample line bundle over X . Assume that X has a regular strictly semistable model \mathcal{X} over R such that L extends to a relatively ample line bundle \mathcal{L} over \mathcal{X} . Let $S(\mathcal{X})$ denote the associated skeleton in X^{an} . Then $S(\mathcal{X})$ has a faithful tropicalization associated to $|L^{\otimes m} \otimes \omega_X|$ for any $m \geq \phi(d)$. More strongly, if ℓ denotes the number of irreducible components of \mathcal{X}_s , then there exist $(\ell + d + 1)$ nonzero global sections of $L^{\otimes m} \otimes \omega_X$ such that the associated morphism $\varphi : X^{\text{an}} \rightarrow \mathbb{TP}^{\ell+d}$ is a faithful tropicalization of $S(\mathcal{X})$.*

Corollary 1.2. *Let R and K be as in Theorem 1.1. Let X be a connected, smooth projective variety of dimension d over K , and assume that X has a regular strictly semistable model \mathcal{X} over R . Let $S(\mathcal{X})$ denote the the associated skeleton in X^{an} .*

- (1) *Let L be an ample line bundle over X , and we assume that L extends to a relatively ample line bundle over \mathcal{X} . Then, if ω_X is trivial (e.g. if X is an abelian variety or a Calabi-Yau manifold) and if $d \leq 4$, then $S(\mathcal{X})$ has a faithful tropicalization associated to $|L^{\otimes m}|$ for any $m \geq d + 1$.*
- (2) *If $\omega_{\mathcal{X}/R}$ is relatively ample and if $d \leq 4$, then $S(\mathcal{X})$ has a faithful tropicalization associated to $|\omega_X^{\otimes m}|$ for any $m \geq d + 2$.*

Faithful tropicalizations have been studied by many authors. Following the one-dimensional case due to Katz–Markwig–Markwig [17, 18], Baker–Payne–Rabinoff [2], and Baker–Rabinoff [3], Gubler–Rabinoff–Werner [14] have shown that for a connected smooth projective variety X of an arbitrary dimension and for any skeleton Γ of X^{an} , there exist nonzero rational functions f_1, \dots, f_n of X such that

$$\psi : X^{\text{an}} \dashrightarrow \mathbb{R}^n, \quad p = (p, |\cdot|) \mapsto (-\log |f_1(p)|, \dots, -\log |f_n(p)|)$$

gives a faithful tropicalization of Γ . A new aspect here is to give a condition on when f_1, \dots, f_n are taken to be global sections of the adjoint line bundle. (See also [19] for related results when $\dim X = 1$).

While the assumption on the existence of a model in Theorem 1.1 may look strong, we remark that, if we replace K with a finite extension field of K and L with a power of L , then we can always achieve this assumption, so that we can apply the theorem; see Remark 2.3.

Mustață–Nicaise [27] and Nicaise–Xu [30] have introduced the essential skeleton $S(X)$ which depends only on X and not on a particular model \mathcal{X} of X by using birational geometry. If X has a regular strictly semistable model of \mathcal{X} , then $S(X)$ is contained in $S(\mathcal{X})$, so that Theorem 1.1 also gives a faithful tropicalization of $S(X)$.

It will be interesting to study a faithful tropicalization of the (more general) skeleton $S(\mathcal{X}, H)$ associated to a strictly semistable pair (cf. [14]) or the skeleton associated to an sncd-model of X (cf. [9], [27]). It may be possible to work over an algebraically closed field which is complete with respect to a non-trivial non-Archimedean value, but working under this setting should be technically more involved (cf. [19]), and we do not pursue it in this paper.

Let us explain our ideas of the proof of Theorem 1.1, together with the structure of this paper. In Section 2, we briefly recall some known facts on Berkovich spaces and tropical geometry. In Section 3, we prove some vanishing of cohomologies and basepoint-freeness on strictly normal crossing varieties, which are technical keys. Then in Section 4, we start the proof of Theorem 1.1. We denote by $\mathcal{X}_s = \bigcup_{1 \leq i \leq \ell} \mathcal{X}_{s,i}$ the irreducible decomposition of the special fiber \mathcal{X}_s of \mathcal{X} . Then each $\mathcal{X}_{s,i}$ corresponds to a point, called a Shilov point (also called a divisorial point) and denoted by $[\mathcal{X}_{s,i}]$, in $S(\mathcal{X})$. We need, for example, to separate $[\mathcal{X}_{s,i}]$ and $[\mathcal{X}_{s,i'}]$ ($i \neq i'$) by global sections of $L^{\otimes m} \otimes \omega_X$. To this end, we first construct suitable global sections η of $\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R}|_{\mathcal{X}_{s,i}}(-W)$ for suitable effective divisors W of $\mathcal{X}_{s,i}$ when $m \geq \phi(d)$. Next, we extend those η 's to global sections of $\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R}|_{\mathcal{X}_{s,i}}$, and using a Kodaira-type vanishing of cohomologies, we extend them to global sections $\tilde{\eta}$ of $\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R}$, where $\omega_{\mathcal{X}/R}$ is the relative dualizing sheaf. Then we restrict to the $\tilde{\eta}$'s to the generic fiber X to obtain global sections s of $L^{\otimes m} \otimes \omega_X$, and we show that those s 's give a local homeomorphism preserving the integral structures between $S(\mathcal{X})$ and its image in tropical projective space, which is called a unimodular tropicalization of $S(\mathcal{X})$. Actually, since we only consider the skeleton $S(\mathcal{X})$ associated to a regular strictly semistable model \mathcal{X} , we show that $(\ell + d + 1)$ nonzero global sections $s_0, s_1, \dots, s_{\ell+d}$, where ℓ is the number of irreducible components of \mathcal{X}_s , are enough to give a unimodular tropicalization. In Section 5, we show somewhat fortunately that this φ is a faithful tropicalization of $S(\mathcal{X})$.

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Notation and conventions. Throughout this paper, let R be a complete discrete valuation ring of equi-characteristic zero with fractional field K and residue field k . Let ϖ denote a uniformizer of R , and let \mathfrak{m} denote the maximal ideal of R . Let $v_K : K \rightarrow \mathbb{Z} \cup \{\infty\}$ denote the additive discrete valuation of K with $v_K(\varpi) = 1$, and let $|\cdot|_K := \exp(-v_K(\cdot))$ denote the multiplicative value of K .

By a *variety*, we mean a reduced separated scheme of finite type over a field, which is allowed to be reducible.

2. PRELIMINARIES

In §§2.1–2.4, we collect some known facts on Berkovich spaces and tropical geometry. Our basic references are [2], [9], [13, §5], [14], [25], [27] and [29]. Similar to [9] and [27], we use the language of schemes (rather than formal schemes), which might be more familiar to readers in birational algebraic geometry. Then, in §2.5, following [2] and [14], we define unimodular and faithful tropicalizations associated to linear systems.

2.1. Strictly semistable model. Let X be a connected, smooth projective variety defined over K of dimension d . By a *strictly semistable model* \mathcal{X} of X , we mean a scheme that is proper and flat over R endowed with an isomorphism $\mathcal{X} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K) \cong X$ such that \mathcal{X} is covered by Zariski open subsets \mathcal{U} that admit an étale morphism

$$(2.3) \quad \psi : \mathcal{U} \rightarrow \mathcal{S} := \mathrm{Spec} R[x_0, \dots, x_d]/(x_0 x_1 \cdots x_r - \varpi),$$

where r is an integer with $0 \leq r \leq d$ depending on \mathcal{U} (cf. [10, (2.16)], [14, Definition 3.1]). Remark that any irreducible component of the special fiber of a strictly semistable model is irreducible. If a strictly semistable model \mathcal{X} is regular, we call \mathcal{X} a *regular* strictly semistable model of X .

We note that X may not admit a regular strictly semistable model over R , but after making a suitable finite extension, one can always find a regular strictly semistable model of X . Indeed, by Hironaka's resolution of singularities (recall that we assume that R has equi-characteristic zero), one can find a proper regular scheme \mathcal{X} over R with generic fiber X for which the reduced closed fiber $\mathcal{X}_{s,\mathrm{red}}$ is a divisor with normal crossings in \mathcal{X} . Then the semistable theorem by Kempf–Knudsen–Mumford–Saint-Donat [21, p. 198] asserts that, after making a finite extension K'/K with discrete valuation ring R' , $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$ admits a regular strictly semistable model over R' .

2.2. Berkovich analytic space. Let X be an irreducible variety defined over K . Then one has X^{an} , the *analytification* of X in the sense of Berkovich (cf. [4], [5], [6], [7]). Although X^{an} has an analytic structure, we recall only how it is described as a topological space. For a scheme point p in X , let $\kappa(p)$ denote the residue field. As a set, X^{an} is given by

$$X^{\mathrm{an}} := \{(p, |\cdot|) \mid p \in X \text{ and } |\cdot| \text{ is an absolute value of } \kappa(p) \text{ extending } |\cdot|_K\}.$$

The underlying topology on X^{an} is the weakest topology such that $\iota : X^{\mathrm{an}} \rightarrow X, (p, |\cdot|) \mapsto p$ is continuous and that for any Zariski open set U of X and for any regular function $g \in \mathcal{O}_X(U)$, the map $\iota^{-1}(U) \rightarrow \mathbb{R}, (p, |\cdot|) \mapsto |g(p)|$ is continuous.

Example 2.1 (classical point). If p is a closed point of X , then $\kappa(p)$ is a finite extension field of the completely valued field K , and hence there exists a unique absolute value $|\cdot|_{\kappa(p)}$ that extends to $|\cdot|_K$. Through the assignment $p \mapsto (p, |\cdot|_{\kappa(p)})$, any closed point of X is regarded as an element of X^{an} .

Example 2.2 (Shilov point). Let $\mathcal{X} \rightarrow \mathrm{Spec}(R)$ be a proper and flat morphism with generic fiber X . Assume that \mathcal{X} is normal. Let E be an irreducible component of the special fiber, and let ξ denote the generic point of E . Then since ξ is a normal point of codimension 1,

$\mathcal{O}_{\mathcal{X},\xi}$ is a discrete valuation ring. Further, $\mathcal{O}_{\mathcal{X},\xi}$ contains R and the fraction field of $\mathcal{O}_{\mathcal{X},\xi}$ equals the function field $K(X)$ of X . It follows that there exists a unique value $|\cdot|_\xi$ on $K(X)$ that is equivalent to the value associated to the discrete valuation ring $\mathcal{O}_{\mathcal{X},\xi}$ and that extends $|\cdot|_K$. Thus, if η denotes the generic point of X , then $(\eta, |\cdot|_\xi) \in X^{\text{an}}$. The point $(\eta, |\cdot|_\xi)$, simply denoted by $[E]$, is called the *Shilov point* associated to (\mathcal{X}, E) . Shilov points are also called divisorial points (cf. [9], [29], for example).

In particular, if X is a connected, smooth projective variety and admits a strictly semistable model \mathcal{X} over R , then each irreducible component of the special fiber \mathcal{X}_s gives a Shilov point in X^{an} .

2.3. Skeleton. Let X be a connected, smooth projective variety defined over K of dimension d , and assume that X admits a regular strictly semistable model \mathcal{X} over R . Then one has the skeleton $S(\mathcal{X}) \subset X^{\text{an}}$ associated to \mathcal{X} , as a special case of [6] (see also [14], [22], [28], [32]). Here we mainly follow [9, §3] and [27, §3] to describe $S(\mathcal{X})$.

Stratification. Let \mathcal{X}_s denote the special fiber of \mathcal{X} . We write $\mathcal{X}_s = \bigcup_{1 \leq i \leq \ell} \mathcal{X}_{s,i}$ for the irreducible decomposition. By the definition of a regular strictly semistable model, \mathcal{X}_s is a reduced divisor of \mathcal{X} , and each $\mathcal{X}_{s,i}$ is a connected, smooth projective variety over k .

We define the stratification of \mathcal{X}_s as follows. We put $\mathcal{X}_s^{(0)} := \mathcal{X}_s$. For each $\alpha \in \mathbb{Z}_{>0}$, let $\mathcal{X}_s^{(\alpha)}$ be the complement of the set of normal points in $\mathcal{X}_s^{(\alpha-1)}$. Thus we obtain a chain of closed subsets:

$$\mathcal{X}_s = \mathcal{X}_s^{(0)} \supsetneq \mathcal{X}_s^{(1)} \supsetneq \cdots \supsetneq \mathcal{X}_s^{(t)} \supsetneq \mathcal{X}_s^{(t+1)} = \emptyset,$$

where $t \leq d$. An irreducible components of $\mathcal{X}_s^{(\alpha)} \setminus \mathcal{X}_s^{(\alpha+1)}$ ($0 \leq \alpha \leq t$) is called a *stratum* of \mathcal{X}_s . A stratum S of \mathcal{X}_s is *minimal* if there does not exist a stratum of \mathcal{X}_s that is strictly contained in the Zariski closure of S .

Canonical simplex associated to a stratum. Let S be a stratum of \mathcal{X} . Then there exist $r \geq 1$ and a subset $J = \{j_1, \dots, j_r\}$ of $\{1, \dots, \ell\}$ such that S is a connected component of $(\bigcap_{j \in J} \mathcal{X}_{s,j}) \setminus \mathcal{X}_s^{(r)}$.

We define the standard $(r-1)$ -simplex Δ^{r-1} by

$$(2.4) \quad \Delta^{r-1} := \{\mathbf{u} := (u_1, \dots, u_r) \in \mathbb{R}^r \mid u_1 \geq 0, \dots, u_r \geq 0, u_1 + \cdots + u_r = 1\}.$$

Let $\text{relint}(\Delta^{r-1}) := \{\mathbf{u} := (u_1, \dots, u_r) \in \mathbb{R}^r \mid u_1 > 0, \dots, u_r > 0, u_1 + \cdots + u_r = 1\}$ denote the relative interior of Δ^{r-1} .

For any $\mathbf{u} \in \Delta^{r-1}$, we are going to define an absolute value $|\cdot|_{\mathbf{u},S}$ on the function field $K(X)$ extending $|\cdot|_K$. Let ξ be the generic point of S . For each $j \in J$, we choose a local equation $T_j = 0$ of $\mathcal{X}_{s,j}$ in \mathcal{X} at ξ . We take any $f \in \mathcal{O}_{\mathcal{X},\xi}$. Let $\widehat{\mathcal{O}}_{\mathcal{X},\xi}$ be the completion of $\mathcal{O}_{\mathcal{X},\xi}$. Since R is of equi-characteristic zero, Cohen's structure theorem asserts that $\widehat{\mathcal{O}}_{\mathcal{X},\xi}$ contains a field isomorphic to the residue field $\kappa(\xi)$ and $\widehat{\mathcal{O}}_{\mathcal{X},\xi}$ is isomorphic to the power series ring $\kappa(\xi)[[T_{j_1}, \dots, T_{j_r}]]$. Then f is written in $\widehat{\mathcal{O}}_{\mathcal{X},\xi} \cong \kappa(\xi)[[T_{j_1}, \dots, T_{j_r}]]$ as

$$f = \sum_{\mathbf{m}=(m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r} a_{\mathbf{m}} T_{j_1}^{m_1} \cdots T_{j_r}^{m_r} \quad (a_{\mathbf{m}} \in \kappa(\xi)),$$

and

$$(2.5) \quad |f|_{\mathbf{u},S} := \max \{ \exp(-u_1 m_1 - \cdots - u_r m_r) \mid \mathbf{m} \in \mathbb{Z}_{\geq 0}^r, a_{\mathbf{m}} \neq 0 \}$$

gives a well-defined absolute value on $\mathcal{O}_{\mathcal{X},\xi}$ (see [27, Proposition 3.1.4]). This absolute value $|\cdot|_{\mathbf{u},S}$ extends to an absolute value $|\cdot|_{\mathbf{u},S}$ on $K(X)$. Further, since $\varpi = \lambda T_{j_1} \cdots T_{j_r}$ for some unit λ in $\mathcal{O}_{\mathcal{X},\xi}$, one has $|\varpi|_{\mathbf{u},S} = \exp(-1)$, which shows that $|\cdot|_{\mathbf{u},S}$ agrees with $|\cdot|_K$ for elements in K . Thus for each \mathbf{u} , we have a point $(\eta, |\cdot|_{\mathbf{u},S}) \in X^{\text{an}}$, where η is the generic point of X .

With the above notation, we set

$$(2.6) \quad \Delta_S := \{(\eta, |\cdot|_{\mathbf{u},S}) \in X^{\text{an}} \mid \mathbf{u} \in \Delta^{r-1}\}$$

and call it the *canonical simplex* associated to S . The assignment

$$(2.7) \quad \Delta^{r-1} \rightarrow \Delta_S, \quad \mathbf{u} \mapsto (\eta, |\cdot|_{\mathbf{u},S})$$

is a homeomorphism, where Δ^{r-1} is endowed with the Euclidean topology (see [27, Proposition 3.1.4]). Via this homeomorphism, we endow Δ_S with a simplex structure. We denote $\text{relint}(\Delta_S)$ the relative interior of the simplex Δ_S . Remark that $\text{relint}(\Delta_S) \cong \text{relint}(\Delta^{r-1})$ under the homeomorphism.

Note that by the construction, $|\cdot|_{\mathbf{u},S}$ is independent of the choice of local equations $T_j = 0$ for $\mathcal{X}_{s,j}$. Thus, up to permutations of $J = \{j_1, \dots, j_r\}$, the homeomorphism between Δ^{r-1} and Δ_S in (2.7) is intrinsic (cf. [14, p. 169]).

Let S' be another stratum of which Zariski closure $\overline{S'}$ contains S . Then $\Delta_{S'}$ is a face of Δ_S , as we now explain. Since $\overline{S'} \supsetneq S$, there exists a nonempty proper subset A of $\{1, \dots, r\}$ such that S' is defined in \mathcal{X} at its generic point by $T_{j_\alpha} = 0$ for all $\alpha \in A$. We set $r' := |A|$. We regard the standard $(r' - 1)$ -simplex $\Delta^{r'-1}$ as the subset of Δ^{r-1} by

$$\Delta^{r'-1} = \{\mathbf{u} = (u_1, \dots, u_r) \in \Delta^{r-1} \mid u_\beta = 0 \text{ for any } \beta \notin A\}.$$

Then, by the definition of $|\cdot|_{\mathbf{u},S}$, we see that $|\cdot|_{\mathbf{u},S} = |\cdot|_{\mathbf{u},S'}$ for any $\mathbf{u} \in \Delta^{r'-1}$. It follows that the homeomorphism $\Delta^{r'-1} \rightarrow \Delta_{S'}$ for S' coincides with the restriction to $\Delta^{r'-1}$ of the homeomorphism $\Delta^{r-1} \rightarrow \Delta_S$ for S .

Skeleton $S(\mathcal{X})$. The *skeleton* $S(\mathcal{X}) \subset X^{\text{an}}$ associated to \mathcal{X} is defined by

$$S(\mathcal{X}) := \bigcup_S \Delta_S,$$

where S runs through all the strata of \mathcal{X}_s .

The description of the canonical simplex associated to a stratum tells us that $S(\mathcal{X})$ is homeomorphic to the *dual intersection complex* of \mathcal{X}_s . The dual intersection complex of \mathcal{X}_s is a simplicial complex whose simplices correspond bijectively to the set of strata of \mathcal{X}_s . To be precise, simplices of dimension α correspond bijectively to the irreducible components of $\mathcal{X}_s^{(\alpha)} \setminus \mathcal{X}_s^{(\alpha+1)}$.

As explained above, if S and S' are strata of \mathcal{X}_s , then the simplex corresponding to S' is a face of the simplex corresponding to S if and only if the Zariski closure of S' contains S . In particular, vertices (i.e., 0-dimensional simplices) of the dual intersection complex correspond bijectively to the irreducible components of \mathcal{X}_s . The assignment of the simplex

corresponding to a stratum S to Δ_S in X^{an} gives the homeomorphism between the dual intersection complex of \mathcal{X}_s and the skeleton $S(\mathcal{X})$. See [22], [9, §3], [27, §3] for details.

We note that $S(\mathcal{X}) = \bigcup_S \Delta_S$, where S runs through all the *minimal* strata of \mathcal{X}_s , and that $S(\mathcal{X}) = \coprod_S \text{relint}(\Delta_S)$, where S runs through all the strata of \mathcal{X}_s .

Remark 2.3. Suppose that we are given a skeleton $S(\mathcal{X})$, where \mathcal{X} is a strictly semistable model of X . Then modulo finite extensions of K , we can faithfully tropicalize $S(\mathcal{X})$ by using Theorem 1.1. Indeed, if we replace K by a suitable finite extension K' with discrete valuation ring R' and L by a suitable power $L^{\otimes a}$, then $X' := X \times_{\text{Spec}(K)} \text{Spec}(K')$ admits a regular strictly semistable model \mathcal{X}' over R' such that $S(\mathcal{X}) \subseteq S(\mathcal{X}')$ (in the Berkovich analytification X'^{an}) and such that $L' := L^{\otimes a} \otimes_K K'$ extends to a relatively ample line bundle over \mathcal{X}' . (Here, we use the same argument as in [9, Corollary 1.5].) This means that modulo finite extensions of K and powers of L , there always exists a model $(\mathcal{X}', \mathcal{L}')$ such that \mathcal{X}' is strictly semistable, \mathcal{L}' is relatively ample and such that $S(\mathcal{X}) \subseteq S(\mathcal{X}')$. Thus Theorem 1.1 gives a faithful tropicalization of $S(\mathcal{X}')$, which means that modulo finite extensions of K , the theorem gives a faithful tropicalization of $S(\mathcal{X})$.

2.4. Tropical projective space and tropicalization map. In this subsection, we recall tropical projective space and tropicalization map from a Berkovich analytic space to tropical projective space.

Integral rational-affine polyhedra. Let N be a free abelian group of rank n , and set $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. By fixing a free basis of N , we identify N with \mathbb{Z}^n and $N_{\mathbb{R}}$ with Euclidean space \mathbb{R}^n . Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of \mathbb{R}^n . An *integral rational-affine polyhedron* in $N_{\mathbb{R}} = \mathbb{R}^n$ is a subset written as $\Delta := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in M_{r,n}(\mathbb{Z})$ and $b \in \mathbb{Q}^r$. The *face* of an integral rational-affine polyhedron Δ is a subset $\{x \in \Delta \mid \langle w, x \rangle \leq \langle w, y \rangle \text{ for all } y \in \Delta\}$ for some $w \in \mathbb{Z}^n$. The relative interior of Δ is denoted by $\text{relint}(\Delta)$. An *integral rational-affine polyhedral complex* Σ is a finite union of integral rational-affine polyhedra such that any face of $\Delta \in \Sigma$ belongs to Σ and such that any nonempty intersection of two polyhedra $\Delta, \Delta' \in \Sigma$ is a face of both Δ and Δ' . An *integral rational-affine map* from \mathbb{R}^n to \mathbb{R}^m is a map of the form $x \mapsto Cx + d$ for some $C \in M_{m,n}(\mathbb{Z})$ and $d \in \mathbb{Q}^m$.

Let $\Delta \subset \mathbb{R}^n$ be an integral rational-affine polyhedron. A map $F : \Delta \rightarrow \mathbb{R}^m$ is an *integral rational-affine map* if F is the restriction of an integral rational-affine map from \mathbb{R}^n to \mathbb{R}^m . Let H be the linear span of $\Delta - x$ for any $x \in \Delta$. Then $H \cap \mathbb{Z}^n$ is a free abelian group of rank $\dim \Delta$. If $F : \Delta \rightarrow \mathbb{R}^m$ is an integral rational-affine map, then $\Delta' := F(\Delta)$ is an integral rational-affine polyhedron in \mathbb{R}^m . Similarly, if H' is the linear span of $\Delta' - x'$ for any $x' \in \Delta'$ then $H' \cap \mathbb{Z}^m$ is a free abelian group of rank $\dim \Delta'$. We say that F is a *unimodular* integral rational-affine map if F is injective and $H' \cap \mathbb{Z}^m = F(H \cap \mathbb{Z}^n)$ (see [14, §2.2]).

Tropical projective space. We set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$. The n -dimensional tropical projective space is defined to be

$$\mathbb{TP}^n := (\mathbb{T}^{n+1} \setminus \{(\infty, \dots, \infty)\}) / \sim,$$

where by definition $x := (x_0, \dots, x_n), y := (y_0, \dots, y_n) \in \mathbb{T}^{n+1} \setminus \{(\infty, \dots, \infty)\}$ satisfy $x \sim y$ if and only if there exists $c \in \mathbb{R}$ such that $y_i = x_i + c$ for all $i = 0, \dots, n$ (see [26]). The equivalence class of x in \mathbb{TP}^n is written as $(x_0 : \dots : x_n)$. Tropical projective space is equipped with $(n+1)$ charts $U_i := \{x = (x_0 : \dots : x_n) \in \mathbb{TP}^n \mid x_i \neq \infty\}$.

Tropicalization map. Let X_0, \dots, X_n be the homogeneous coordinates of \mathbb{P}^n . Let $\mathbb{P}^{n,\text{an}}$ denote the Berkovich analytification of \mathbb{P}^n . Then we have the tropicalization map

$$(2.8) \quad \text{trop} : \mathbb{P}^{n,\text{an}} \rightarrow \mathbb{TP}^n, \quad p = (p, |\cdot|) \mapsto (-\log |X_0(p)| : \dots : -\log |X_n(p)|).$$

Since we fix the homogeneous coordinates, the multiplicative group \mathbb{G}_m^n is embedded in \mathbb{P}^n , where $\mathbb{G}_m^n = \{(1 : x_1 : \dots : x_n) \mid x_1 \neq 0, \dots, x_n \neq 0\}$. Similarly, the Euclidean space \mathbb{R}^n is embedded in \mathbb{TP}^n , where $\mathbb{R}^n = \{(0 : x_1 : \dots : x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$. Then the restriction of the tropicalization map to \mathbb{G}_m^n gives $\text{trop}|_{\mathbb{G}_m^n} : \mathbb{G}_m^n \rightarrow \mathbb{R}^n$, and $\text{trop}^{-1}(\mathbb{R}^n) = \mathbb{G}_m^n$.

Tropical geometry near the boundary $\mathbb{TP}^n \setminus \mathbb{R}^n$ is rather subtle (see [26]), but for our purposes (i.e., for faithful tropicalizations), we will not need analysis on the boundary.

Let Y° be an irreducible variety over K , and assume that Y° is embedded in \mathbb{G}_m^n as a closed subvariety. The *tropicalization* of Y° is the subset $\text{trop}(Y^{\circ,\text{an}})$ in \mathbb{R}^n . By the Bieri–Groves theorem [8], $\text{trop}(Y^{\circ,\text{an}})$ is the support of an integral rational-affine polyhedral complex Σ in \mathbb{R}^n .

2.5. Unimodular tropicalization and faithful tropicalization. Let X be a connected, smooth projective variety over K , and let L be an ample line bundle over X . Assume that X has a regular strictly semistable model \mathcal{X} of X over R . Let $S(\mathcal{X}) \subset X^{\text{an}}$ be the skeleton associated to \mathcal{X} .

We have $S(\mathcal{X}) = \bigcup_S \Delta_S$, where S runs through all the strata of \mathcal{X}_s . Recall that Δ_S is homeomorphic to Δ^{r-1} for some r by (2.7) and that this homeomorphism is intrinsic in the sense that it is unique up to reordering of the coordinates. Through this homeomorphism, we will regard Δ_S as an integral rational-affine polyhedron in \mathbb{R}^r .

Suppose that we are given nonzero global sections $s_0, s_1, \dots, s_n \in H^0(X, L)$ such that

$$(2.9) \quad \varphi : X^{\text{an}} \longrightarrow \mathbb{TP}^n, \quad p = (p, |\cdot|) \mapsto (-\log |s_0(p)| : \dots : -\log |s_n(p)|)$$

is a morphism (i.e., everywhere defined). Since X^{an} contains closed points of X as classical points (cf. Example 2.1), it follows that $\varphi' : X \longrightarrow \mathbb{P}^n$ defined by $p \mapsto (s_0(p) : \dots : s_n(p))$ is a morphism. We denote by the same φ' the induced morphism from X^{an} to $\mathbb{P}^{n,\text{an}}$. Then we have $\varphi = \text{trop} \circ \varphi'$, where $\text{trop} : \mathbb{P}^{n,\text{an}} \rightarrow \mathbb{TP}^n$ is the tropicalization map in (2.8).

Remark 2.4. With the above notation, we further set $f_i := s_i/s_0$ for $i = 1, \dots, n$. Then f_i is a nonzero rational function on X . Since any point in $S(\mathcal{X})$ is an absolute value on the function field of X , we have $|f_i(p)| \neq 0$ for any $p \in S(\mathcal{X})$. It follows that the restriction of φ defined in (2.9) to $S(\mathcal{X})$ factors through

$$\varphi|_{S(\mathcal{X})} : S(\mathcal{X}) \rightarrow \mathbb{R}^n, \quad p = (p, |\cdot|) \mapsto (-\log |f_1(p)|, \dots, -\log |f_n(p)|)$$

and the standard embedding $\mathbb{R}^n \hookrightarrow \mathbb{TP}^n$, $(x_1, \dots, x_n) \mapsto (0 : x_1 : \dots : x_n)$ in §2.4. In the following, we sometimes regard $\varphi|_{S(\mathcal{X})}$ as a map valued in \mathbb{R}^n .

We put $Y^\circ := \varphi'(X) \cap \mathbb{G}_m^n \subset \mathbb{P}^n$. Then $\text{trop}(Y^{\circ,\text{an}})$ is the support of some integral rational-affine polyhedral complex Σ in \mathbb{R}^n . For $p \in S(\mathcal{X})$, we have $|s_i(p)| \neq 0$ for any i , so that $\varphi'(p) \in Y^{\circ,\text{an}}$. Thus we have $\varphi|_{S(\mathcal{X})} : S(\mathcal{X}) \rightarrow \Sigma \subset \mathbb{R}^n$.

Definition 2.5 (cf. [2, (5.15)], [14, Definition 9.2]). Let $s_0, s_1, \dots, s_n \in H^0(X, L)$ be nonzero global sections of L such that $\varphi : X^{\text{an}} \longrightarrow \mathbb{TP}^n$ in (2.9) is a morphism.

- (1) (unimodular tropicalization) Let S be a stratum of \mathcal{X}_s . We say that φ is *unimodular* on Δ_S if Δ_S can be covered by finitely many integral rational-affine polyhedra Δ such that $\varphi|_{\Delta}$ is a unimodular integral rational-affine map on Δ (cf. §2.4). We say that φ is a *unimodular tropicalization* of $S(\mathcal{X})$ if φ is unimodular on Δ_S for any stratum S of \mathcal{X}_s .
- (2) (faithful tropicalization) We say that φ is a *faithful tropicalization* of $S(\mathcal{X})$ if φ is a unimodular tropicalization of $S(\mathcal{X})$ such that the restriction of φ to $S(\mathcal{X})$ is injective.

We say that the skeleton $S(\mathcal{X})$ has a *unimodular (resp. faithful) tropicalization associated to the linear system $|L|$* if there exist nonzero global sections $s_0, s_1, \dots, s_n \in H^0(X, L)$ such that the associated morphism φ gives a unimodular (resp. faithful) tropicalization of $S(\mathcal{X})$.

Remark 2.6. For the definitions of unimodular and faithful tropicalizations in Definition 2.5, we require that φ in (2.9) be everywhere defined. This requirement could be relaxed as long as $\varphi|_{S(\mathcal{X})}$ is everywhere defined. Suppose that we are given nonzero global sections $s_0, s_1, \dots, s_n \in H^0(X, L)$, and we set

$$(2.10) \quad \varphi : X^{\text{an}} \dashrightarrow \mathbb{TP}^n, \quad p = (p, |\cdot|) \mapsto (-\log |s_0(p)| : \dots : -\log |s_n(p)|),$$

which may not be defined at some classical points (cf. Example 2.1). As we argue in Remark 2.4, $\varphi|_{S(\mathcal{X})} : S(\mathcal{X}) \rightarrow \mathbb{TP}^n$ is everywhere defined. We call φ in (2.10) a *rational unimodular (resp. faithful) tropicalization* of $S(\mathcal{X})$ if $\varphi|_{S(\mathcal{X})}$ satisfies the condition (1) (resp. (2)) of Definition 2.5, and, in this case, we say that $S(\mathcal{X})$ has a *rational unimodular (resp. faithful) tropicalization associated to the linear system $|L|$* .

Since X^{an} contains closed points of X as classical points, we see that $S(\mathcal{X})$ has a unimodular (resp. faithful) tropicalization associated to $|L|$ if and only if (i) $S(\mathcal{X})$ has a rational unimodular (resp. faithful) tropicalization associated to $|L|$ and (ii) L is basepoint free.

3. VANISHING AND BASEPOINT-FREENESS

In this section, we give some remarks on vanishing of cohomologies and basepoint-freeness over reducible varieties.

First, we recall a Kodaira-type vanishing theorem for a projective strictly normal crossing variety. Here a *strictly* normal crossing variety is a normal crossing variety such that each irreducible component is smooth.

Proposition 3.1. *Let k be a field of characteristic 0. Let Z be a projective strictly normal crossing variety over k . Let N be an ample line bundle over X , and let ω_Z denote the canonical line bundle of Z . Then we have $H^i(Z, N \otimes \omega_Z) = 0$ for any $i > 0$.*

Proof. By a flat base change, we may assume that k is algebraically closed. The assertion follows from, for example, [12, Theorem 1.1] (applied to $Y := \mathcal{X}_s, \Delta := 0$ and $X = \text{Spec}(k)$ in *op. cit.*). Or, since normal crossing varieties are Cohen-Macaulay, the assertion also follows from [23, Corollary 6.6], where the authors of [23] prove the vanishing of the dual $H^i(Z, \omega_Z^{-1}) = 0$ for any $i < \dim Z$. \square

Next, we consider basepoint freeness on strictly normal crossing varieties. For a positive integer d , we consider the following quantity related to the adjoint line bundle over *strictly*

normal crossing varieties in characteristic zero:

$$(3.11) \quad \tilde{\phi}(d) := \left\{ m_0 \in \mathbb{Z} \left| \begin{array}{l} \text{For any projective strictly normal crossing variety } Z \text{ with} \\ \dim Z \leq d \text{ over a field of characteristic zero, and for any} \\ \text{ample line bundle } N \text{ over } Z, N^{\otimes m} \otimes \omega_Z \text{ is basepoint free} \\ \text{for any } m \geq m_0. \end{array} \right. \right\}.$$

Compared with $\phi(d)$ in (1.2), we allow that Z is a projective strictly normal crossing variety; Z is not necessarily smooth. By convention, if there does not exist such an m_0 , then we set $\tilde{\phi}(d) := +\infty$, but this does not occur as we see below.

Lemma 3.2. *We have $\phi(d) = \tilde{\phi}(d)$.*

Proof. It suffices to show that $\tilde{\phi}(d) \leq \phi(d)$. Let Z be a projective strictly normal crossing variety of dimension less than or equal to d over a field k of characteristic zero, let N be an ample line bundle over Z , and let $m \geq \phi(d)$. We are going to show that $N^{\otimes m} \otimes \omega_Z$ is basepoint free by induction on $\dim Z$. By a flat base change, we may assume that k is algebraically closed. Also we may assume that Z is connected.

We put $\dim Z = e (\leq d)$. If $e = 0$, then any line bundle is basepoint free, and there is nothing to prove. Let $e \geq 1$, and let $Z = \bigcup_{i=1}^{\ell} Z_i$ be the irreducible decomposition of Z . We take any closed point p of Z . Without loss of generality, we may assume that p lies on Z_1 . We set $Z'_1 := \bigcup_{i=2}^{\ell} Z_i$, and $W_1 := Z_1 \cap Z'_1$. Note that W_1 is a projective strictly normal crossing variety of dimension $e - 1$.

Case 1. Suppose that p is a smooth point of Z_1 . Since $N^{\otimes m} \otimes \omega_Z|_{Z_1}(-W_1) = N|_{Z_1}^{\otimes m} \otimes \omega_{Z_1}$, by the definition of $\phi(d)$, there exists $t \in H^0(Z_1, N^{\otimes m} \otimes \omega_Z|_{Z_1}(-W_1))$ with $t(p) \neq 0$. Via the natural injection $N^{\otimes m} \otimes \omega_Z|_{Z_1}(-W_1) \hookrightarrow N^{\otimes m} \otimes \omega_Z|_{Z_1}$, let $s \in H^0(Z_1, N^{\otimes m} \otimes \omega_Z|_{Z_1})$ be the image of t . Then $s(p) \neq 0$ and s is equal to zero along W_1 . Let $\tilde{s} \in H^0(Z, N^{\otimes m} \otimes \omega_Z)$ be the zero extension of s to Z , that is, $\tilde{s}|_{Z_1} = s$ and \tilde{s} vanishes outside of Z_1 . Then $\tilde{s}(p) \neq 0$, which shows that $N^{\otimes m} \otimes \omega_Z$ is free at p .

Case 2. Suppose that p lies on W_1 . By the induction hypothesis, there exists $s \in H^0(W_1, N|_{W_1}^{\otimes m} \otimes \omega_{W_1})$ such that $s(p) \neq 0$. By the adjunction formula, there exists a natural exact sequence

$$0 \rightarrow N|_{Z_1}^{\otimes m} \otimes \omega_{Z_1} \rightarrow N|_{Z_1}^{\otimes m} \otimes \omega_{Z_1}(W_1) \rightarrow N|_{W_1}^{\otimes m} \otimes \omega_{W_1} \rightarrow 0,$$

and by the Kodaira vanishing we have $h^1(Z_1, N|_{Z_1}^{\otimes m} \otimes \omega_{Z_1}) = 0$. It follows that there exists $s_1 \in H^0(Z_1, N|_{Z_1}^{\otimes m} \otimes \omega_{Z_1}(W_1))$ with $s_1|_{W_1} = s$. Similarly, we have an exact sequence

$$0 \rightarrow N|_{Z'_1}^{\otimes m} \otimes \omega_{Z'_1} \rightarrow N|_{Z'_1}^{\otimes m} \otimes \omega_{Z'_1}(W_1) \rightarrow N|_{W_1}^{\otimes m} \otimes \omega_{W_1} \rightarrow 0$$

by the adjunction formula and the vanishing $h^1(Z'_1, N|_{Z'_1} \otimes \omega_{Z'_1}) = 0$ by Proposition 3.1, so that there exists $s'_1 \in H^0(Z'_1, N|_{Z'_1} \otimes \omega_{Z'_1}(W_1))$ with $s'_1|_{W_1} = s$. Since

$$N|_{Z_1}^{\otimes m} \otimes \omega_{Z_1}(W_1) \cong (N^{\otimes m} \otimes \omega_Z)|_{Z_1} \quad \text{and} \quad N|_{Z'_1}^{\otimes m} \otimes \omega_{Z'_1}(W_1) \cong (N^{\otimes m} \otimes \omega_Z)|_{Z'_1},$$

we regard s_1 and s'_1 as sections of $H^0(Z_1, (N^{\otimes m} \otimes \omega_Z)|_{Z_1})$ and $H^0(Z'_1, (N^{\otimes m} \otimes \omega_Z)|_{Z'_1})$, respectively. Since $s_1|_{W_1} = s'_1|_{W_1}$, s_1 and s'_1 glue together to give a section $\tilde{s} \in H^0(Z, N^{\otimes m} \otimes \omega_Z)$. Then $\tilde{s}(p) = s(p) \neq 0$, which shows that $N^{\otimes m} \otimes \omega_Z$ is free at p . \square

4. UNIMODULAR TROPICALIZATION

As before, let X be a connected, smooth projective variety of dimension d over K . Let L be an ample line bundle over X . Let \mathcal{X} be a regular strictly semistable model of X over R . Let \mathcal{L} be a line bundle over \mathcal{X} with $\mathcal{L}|_X = L$ and assume that \mathcal{L} is relatively ample.

In this section, under the assumption in Theorem 1.1, we show that there exist global sections s_0, s_1, \dots, s_n of $L^{\otimes m} \otimes \omega_X$ that give a *unimodular* tropicalization of the skeleton $S(\mathcal{X})$.

We prove two technical lemmas.

Lemma 4.1. *Under the assumption of Theorem 1.1, the restriction map $H^0(\mathcal{X}, \mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R}) \rightarrow H^0(\mathcal{X}_s, \mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R}|_{\mathcal{X}_s})$ is surjective.*

Proof. We note that $\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R}|_{\mathcal{X}_s} = (\mathcal{L}|_{\mathcal{X}_s})^{\otimes m} \otimes \omega_{\mathcal{X}_s}$. By the base change theorem (see [15, III, 12]), it then suffices to show that $h^1(\mathcal{X}_s, (\mathcal{L}|_{\mathcal{X}_s})^{\otimes m} \otimes \omega_{\mathcal{X}_s}) = 0$. Since \mathcal{X}_s is a strictly normal crossing variety and $\mathcal{L}|_{\mathcal{X}_s}$ is an ample line bundle over \mathcal{X}_s , the vanishing $h^1(\mathcal{X}_s, (\mathcal{L}|_{\mathcal{X}_s})^{\otimes m} \otimes \omega_{\mathcal{X}_s}) = 0$ follows from Proposition 3.1. \square

We fix the notation. Let $\mathcal{X}_s = \bigcup_{1 \leq i \leq \ell} \mathcal{X}_{s,i}$ denote the irreducible decomposition of the special fiber \mathcal{X}_s . For each i ($1 \leq i \leq \ell$), let $\bigcup_{j=1}^{b_i} Y_{ij}$ denote the irreducible decomposition of $\mathcal{X}_{s,i} \cap (\mathcal{X}_s - \mathcal{X}_{s,i}) = \mathcal{X}_{s,i} \cap \bigcup_{1 \leq i' \leq \ell, i' \neq i} \mathcal{X}_{s,i'}$.

Recall that $\phi(d)$ is a positive integer defined in (1.2).

Lemma 4.2. *Under the assumption of Theorem 1.1, $(\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_{s,i}} \left(-\sum_{j=1}^{b_i} Y_{ij} \right)$ is basepoint free for any $m \geq \phi(d)$.*

Proof. By the adjunction formula, we get

$$\begin{aligned} (\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_{s,i}} \left(-\sum_{j=1}^{b_i} Y_{ij} \right) &\cong (\mathcal{L}|_{\mathcal{X}_{s,i}})^{\otimes m} \otimes \omega_{\mathcal{X}_{s,i}} \left(-\sum_{j=1}^{b_i} Y_{ij} \right) \\ &\cong (\mathcal{L}|_{\mathcal{X}_{s,i}})^{\otimes m} \otimes \omega_{\mathcal{X}_{s,i}}. \end{aligned}$$

Since \mathcal{L} is assumed to be ample, $\mathcal{L}|_{\mathcal{X}_{s,i}}$ is ample. By the definition of $\phi(d)$, $(\mathcal{L}|_{\mathcal{X}_{s,i}})^{\otimes m} \otimes \omega_{\mathcal{X}_{s,i}}$ is basepoint free for any $m \geq \phi(d)$. Thus $(\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_{s,i}} \left(-\sum_{j=1}^{b_i} Y_{ij} \right)$ is basepoint free for any $m \geq \phi(d)$. \square

Well-behaved global sections. By Lemma 4.2, there exists a non-zero global section

$$\xi_i \in H^0 \left(\mathcal{X}_{s,i}, (\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_{s,i}} \left(-\sum_{j=1}^{b_i} Y_{ij} \right) \right)$$

that does not vanish at the generic point of any minimal stratum of $\mathcal{X}_{s,i}$. We denote by $\eta'_i \in H^0 \left(\mathcal{X}_{s,i}, (\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_{s,i}} \right)$ the image of ξ_i under the natural inclusion

$$(\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_{s,i}} \left(-\sum_{j=1}^{b_i} Y_{ij} \right) \hookrightarrow (\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_{s,i}}.$$

We then have $\text{ord}_{Y_{ij}}(\eta'_i) = 1$ for any $j = 1, \dots, b_i$. It follows that there exists a global section η_i of $(\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_s}$ that is the zero-extension of η'_i , that is, $\eta_i|_{\mathcal{X}_{s,i}} = \eta'_i$ and $\eta_i|_{\mathcal{X}_{s,i'}} = 0$ for $i' \neq i$. By Lemma 4.1, there exists $\tilde{\eta}_i \in H^0(\mathcal{X}, \mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})$ such that $\tilde{\eta}_i|_{\mathcal{X}_s} = \eta_i$.

Let $\mathcal{X}_{s,i'} \in \text{Irr}(\mathcal{X}_s)$ ($i' \neq i$) be an irreducible component of \mathcal{X}_s such that $\mathcal{X}_{s,i} \cap \mathcal{X}_{s,i'} \neq \emptyset$. Since η_i is the zero extension of η'_i , we have $\text{ord}_{\mathcal{X}_{s,i'}}(\tilde{\eta}_i) \geq 1$. We claim that

$$(4.12) \quad \text{ord}_{\mathcal{X}_{s,i'}}(\tilde{\eta}_i) = 1.$$

Indeed, let $Y_{ii'}$ be an irreducible component of $\mathcal{X}_{s,i} \cap \mathcal{X}_{s,i'}$. Let \mathcal{V} be a small Zariski open neighborhood of the generic point of $Y_{ii'}$ in \mathcal{X} . Let $T_{i'} = 0$ be a local equation of $\mathcal{X}_{s,i'}$ on \mathcal{V} . We write $\tilde{\eta}_i|_{\mathcal{V}} = T_{i'}^a f_{i'}$ with $a \geq 0$ and $f_{i'}|_{\mathcal{X}_{s,i'}} \not\equiv 0$ on \mathcal{V} . Since $\tilde{\eta}_i|_{\mathcal{X}_{s,i'}} = 0$, we have $a \geq 1$. On the other hand, since $\tilde{\eta}_i|_{\mathcal{X}_{s,i}} = T_{i'}^a|_{\mathcal{X}_{s,i}} f_{i'}|_{\mathcal{X}_{s,i}}$, we have $\text{ord}_{Y_{ii'}}(\tilde{\eta}_i|_{\mathcal{X}_{s,i}}) \geq a$. Since $\text{ord}_{Y_{ii'}}(\tilde{\eta}_i|_{\mathcal{X}_{s,i}}) = \text{ord}_{Y_{ii'}}(\eta'_i) = 1$, we obtain $1 \geq a$. Thus $a = 1$, and we obtain (4.12).

We set

$$(4.13) \quad s_i := \tilde{\eta}_i|_X \in H^0(X, L^{\otimes m} \otimes \omega_X).$$

Base global section. Since $m \geq \phi(d)$ and since $\phi(d) = \tilde{\phi}(d)$ by Lemma 3.2, $(\mathcal{L}|_{\mathcal{X}_s})^{\otimes m} \otimes \omega_{\mathcal{X}_s} = (\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_s}$ is basepoint free. It follows that there exists a global section

$$\eta_0 \in H^0\left(\mathcal{X}_s, (\mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})|_{\mathcal{X}_s}\right)$$

such that η_0 does not vanish at the generic point of any minimal stratum of \mathcal{X}_s . By Lemma 4.1, there exists $\tilde{\eta}_0 \in H^0(\mathcal{X}, \mathcal{L}^{\otimes m} \otimes \omega_{\mathcal{X}/R})$ such that $\tilde{\eta}_0|_{\mathcal{X}_s} = \eta_0$. We set

$$(4.14) \quad s_0 := \tilde{\eta}_0|_X \in H^0(X, L^{\otimes m} \otimes \omega_X).$$

Unimodular tropicalization. Let s_0, s_1, \dots, s_ℓ be global sections of $L^{\otimes m} \otimes \omega_X$ constructed in (4.13) and (4.14). Since $m \geq \phi(d)$, the definition of $\phi(d)$ tells us that there exist global sections $s_{\ell+1}, \dots, s_{\ell+d+1}$ of $L^{\otimes m} \otimes \omega_X$ with $\bigcap_{i=1}^{\ell+d+1} \text{div}(s_i) = \emptyset$. Then $X \rightarrow \mathbb{P}^{\ell+d+1}$ given by $p \mapsto (s_0(p) : \dots : s_{\ell+d+1}(p))$ is a morphism. Let

$$(4.15) \quad \varphi : X^{\text{an}} \rightarrow \mathbb{TP}^{\ell+d+1}, \quad p = (p, |\cdot|) \mapsto (-\log |s_0(p)| : \dots : -\log |s_{\ell+d+1}(p)|)$$

be the associated morphism. For $i = 1, \dots, \ell$, we define the rational function $f_i \in \text{Rat}(X) = \text{Rat}(\mathcal{X})$ by

$$f_i := \frac{s_i}{s_0} = \frac{\tilde{\eta}_i}{\tilde{\eta}_0} \Big|_X.$$

As noted in Remark 2.4, we identify $\varphi|_{S(\mathcal{X})}$ with

$$(4.16) \quad \varphi|_{S(\mathcal{X})} : S(\mathcal{X}) \rightarrow \mathbb{R}^{\ell+d+1}, \quad p \mapsto (-\log |f_1(p)|, \dots, -\log |f_{\ell+d+1}(p)|).$$

Let us prove that the map in (4.16) is a unimodular tropicalization for $S(\mathcal{X})$. We take any minimal stratum S of \mathcal{X}_s . Let Δ_S be the canonical simplex associated to S in $S(\mathcal{X})$. By the construction of the skeleton $S(\mathcal{X})$, it suffices to check that $\varphi|_{\Delta_S}$ is a unimodular integral rational-affine map on Δ_S . Let v_{j_1}, \dots, v_{j_r} be the vertices of Δ_S , where v_j is the Shilov point of $\mathcal{X}_{s,j}$. We set $J := \{j_1, \dots, j_r\} \subset \{1, \dots, \ell\}$. We take a sufficiently small Zariski open neighborhood \mathcal{U} of the generic point of S in \mathcal{X} . Then, since $\tilde{\eta}_0$ does not

vanish at any minimal stratum and hence does not at any stratum, we have $\operatorname{div}(\tilde{\eta}_0)|_{\mathcal{U}} = 0$. Further, for $j \in J$, by (4.12), we have

$$\operatorname{div}(\tilde{\eta}_j)|_{\mathcal{U}} = \sum_{\substack{j' \in J \\ j' \neq j}} \mathcal{X}_{s,j'}.$$

It follows that, regarding f_j as an element of $\operatorname{Rat}(\mathcal{X})$, we have

$$(4.17) \quad \operatorname{div}(f_j)|_{\mathcal{U}} = \sum_{\substack{j' \in J \\ j' \neq j}} \mathcal{X}_{s,j'}.$$

We consider the morphism

$$(4.18) \quad \psi_S : \Delta_S \rightarrow \mathbb{R}^r, \quad p = (p, |\cdot|) \mapsto (-\log |f_{j_1}(p)|, \dots, -\log |f_{j_r}(p)|).$$

We study the function $-\log |f_{j_a}|$ for $a = 1, \dots, r$ in more detail. For $j_a \in J$, let $T_{j_a} = 0$ be a local equation of \mathcal{X}_{s,j_a} at the generic point of S . By (4.17), we write $f_{j_a} = \lambda \prod_{1 \leq b \leq r, b \neq a} T_{j_b}$ for some invertible function λ on \mathcal{U} . We take any $p = (\eta, |\cdot|_{\mathbf{u},S}) \in \Delta_S$, with the notation in (2.6). We write $\mathbf{u} = (u_1, \dots, u_r)$. Then by the definition of $|\cdot|_{\mathbf{u},S}$, we get

$$-\log |f_{j_a}(p)| = \sum_{1 \leq b \leq r, b \neq a} u_b.$$

Since p corresponds to $\mathbf{u} = (u_1, \dots, u_r)$ via the isomorphism $\Delta_S \cong \Delta^{r-1}$ of simplices, we see in particular that $-\log |f_{j_a}(p)|$ is an affine function on Δ_S . Thus ψ_S is an affine map.

Put $\mathbf{v}_a = \psi_S(v_{j_a})$ for $a = 1, \dots, r$. Then $\psi_S(\Delta_S)$ is the $(r-1)$ -dimensional simplex spanned by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$. To show that $\psi_S : \Delta_S \rightarrow \mathbb{R}^r$ is a unimodular integral rational-affine map, it suffices to verify that $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_r - \mathbf{v}_1$ is a part of a \mathbb{Z} -basis of $\mathbb{Z}^r \subset \mathbb{R}^r$. Put $\mathbf{1} = (1, 1, \dots, 1)$, and $\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_r = (0, 0, \dots, 1)$ in \mathbb{Z}^r . Since $\mathbf{v}_a = \mathbf{1} - \mathbf{e}_a$ for $a = 1, \dots, r$, $\mathbf{v}_a - \mathbf{v}_1 = \mathbf{e}_1 - \mathbf{e}_a$ for $a = 2, \dots, r$. Since $\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \dots, \mathbf{e}_1 - \mathbf{e}_r$ is a \mathbb{Z} -basis of \mathbb{Z}^r , it follows that $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_r - \mathbf{v}_1$ is a part of a \mathbb{Z} -basis of \mathbb{Z}^r . Thus we have shown that ψ_S in (4.18) is unimodular on Δ_S . Since ψ_S factors through $\varphi|_{\Delta_S}$, transitivity of lattice indices show that φ in (4.16) is unimodular on Δ_S (see [2, Lemma 5.17] and [14, Lemma 9.3]). This concludes that φ is a unimodular tropicalization of $S(\mathcal{X})$.

Remark 4.3. In the above, we have shown that ψ_S in (4.18) is affine and unimodular. It follows that ψ_S is injective. Since ψ_S factors through $\varphi|_{\Delta_S}$, this shows that $\varphi|_{\Delta_S}$ is injective.

We put together some properties of $-\log |f_{j_a}|$ on $S(\mathcal{X})$, which we use in the next section.

Lemma 4.4. *As above, let v_{j_1}, \dots, v_{j_r} be the vertices of Δ_S . For $a = 1, \dots, r$, we set $g := -\log |f_{j_a}|$. Then g is affine on Δ_S . Further, for any vertex v of $S(\mathcal{X})$, we have*

$$g(v) \begin{cases} = 0 & (\text{if } v = v_{j_a}), \\ = 1 & (\text{if } v = v_{j_b} \text{ for some } b = 1, \dots, r \text{ with } b \neq a). \\ \geq 1 & (\text{otherwise}). \end{cases}$$

Proof. We have already seen that g is affine on Δ_S . Recall that $f_{j_a} = \tilde{\eta}_{j_a}/\tilde{\eta}_0$ as a rational function on \mathcal{X} and that $\tilde{\eta}_0$ does not vanish at the generic point of any stratum. Recall also that v_{j_a} is the Shilov point of \mathcal{X}_{s,j_a} . Since $\tilde{\eta}_{j_a}$ does not vanish at the generic point of

\mathcal{X}_{s,j_a} , the first equality follows. The second equality follows from (4.12). Further, the third inequality follows from the fact that $\tilde{\eta}_{j_a}|_{\mathcal{X}_{s,i'}} = 0$ for $i' \neq i$. \square

5. FAITHFUL TROPICALIZATION

In this section, we show that the unimodular tropicalization of $S(\mathcal{X})$ constructed in (4.15) is actually a faithful tropicalization. We only need to show that φ is *injective*.

We keep the notation in §4. We begin with some lemmas.

Lemma 5.1. *Let S be a stratum of \mathcal{X}_s . Let ϕ be a non-zero rational function on \mathcal{X} . Let $\Delta_S \subset X^{\text{an}}$ be the canonical simplex associated to S , and we set $g := -\log |\phi| : \Delta_S \rightarrow \mathbb{R}$. Suppose that there exist a Zariski open neighborhood \mathcal{U} in \mathcal{X} of the generic point of S and a vertical divisor V on \mathcal{U} such that $\text{div}(\phi|_{\mathcal{U}}) - V$ is a horizontal effective divisor on \mathcal{U} . Let v_{j_1}, \dots, v_{j_r} be the vertices of Δ_S , where v_j is the Shilov point of $\mathcal{X}_{s,j}$. For $\mathbf{u} = (u_1, \dots, u_r) \in \Delta^{r-1}$, we take the point $p = (\eta, |\cdot|_{\mathbf{u},S}) \in \Delta_S$. Then, we have*

$$(5.19) \quad g(p) \geq u_1 g(v_{j_1}) + \dots + u_r g(v_{j_r}).$$

Proof. As before, let $T_j = 0$ be a local equation of $\mathcal{X}_{s,j}$ on \mathcal{U} . Shrinking \mathcal{U} if necessary, we write $V = \sum_{a=1}^r m_a \mathcal{X}_{s,j_a}$, and set $\phi' = T_{j_1}^{-m_1} \dots T_{j_r}^{-m_r}$, so that $\text{div}(\phi') = -V$ on \mathcal{U} . Then on \mathcal{U} we have

$$-\log |\phi'| = m_1 (\log |T_{j_1}|) + \dots + m_r (\log |T_{j_r}|).$$

Since $\log |T_{j_1}|, \dots, \log |T_{j_r}|$ are affine functions on Δ_S , we see that $h := -\log |\phi'|$ is an affine function on Δ_S .

By assumption, $\text{div}(\phi\phi') = \text{div}(\phi) - V$ is a horizontal effective divisor on \mathcal{U} . Set $g' := -\log |\phi\phi'|$. Then $g' = g + h$. Since h is affine,

$$h(p) = u_1 h(v_{j_1}) + \dots + u_r h(v_{j_r}).$$

This means that if we show the inequality for g' in place of g in (5.19), then the same inequality holds also for g . Thus we may and do assume that $\text{div}(\phi)$ is a horizontal effective divisor on \mathcal{U} .

Since $\text{ord}_{\mathcal{X}_{s,j_a}}(\phi) = 0$, we get $g(v_{j_a}) = 0$ for any $a = 1, \dots, r$. Thus it suffices to show that $g(p) \geq 0$. Let ξ denote the generic point of S . From the construction of $|\cdot|_{\mathbf{u},S}$ in (2.5), if $f \in \mathcal{O}_{\mathcal{X},\xi}$, then $|f|_{\mathbf{u},S} \leq 1$. Since ϕ is a regular function on \mathcal{U} and \mathcal{U} contains ξ , we see that $\phi \in \mathcal{O}_{\mathcal{X},\xi}$. We then have $|\phi|_{\mathbf{u},S} \leq 1$. This proves $g(p) \geq 0$. \square

Theorem 5.2. *The map φ in (4.15) is a faithful tropicalization of $S(\mathcal{X})$.*

Proof. Since we have shown that φ is a unimodular tropicalization of $S(\mathcal{X})$ in the previous section, we have only to show that φ is injective.

We have $S(\mathcal{X}) = \coprod_S \text{relint}(\Delta_S)$, where S runs through all the strata of \mathcal{X}_s . As noted in Remark 4.3, φ is injective on each canonical simplex. Thus it suffices to show that $\varphi(\text{relint}(\Delta_S)) \cap \varphi(\text{relint}(\Delta_T)) = \emptyset$ for any strata S and T with $S \neq T$. Furthermore, since it is injective on each canonical simplex, we may and do assume that $\Delta_T \not\subseteq \Delta_S$ and $\Delta_S \not\subseteq \Delta_T$. By symmetry, we assume that $\Delta_T \not\subseteq \Delta_S$. As before, let v_{j_1}, \dots, v_{j_r} be the vertices of Δ_S , where v_j is the Shilov point of $\mathcal{X}_{s,j}$. Since $\Delta_T \not\subseteq \Delta_S$, there exists $1 \leq a \leq r$ such that $v_{j_a} \notin \Delta_T$.

We put $f_j := s_j/s_0$ for all j . Recall from Remark 2.4 that φ is identified with the map

$$S(\mathcal{X}) \rightarrow \mathbb{R}^{\ell+d+1}, \quad p = (p, |\cdot|) \mapsto (-\log |f_1(p)|, \dots, -\log |f_{\ell+d+1}(p)|).$$

We consider the function

$$g : S(\mathcal{X}) \rightarrow \mathbb{R}, \quad p = (p, |\cdot|) \mapsto -\log |f_{j_a}(p)|.$$

It suffices to prove that $g(\text{relint}(\Delta_S)) \cap g(\text{relint}(\Delta_T)) = \emptyset$. First, we take any $p \in \text{relint}(\Delta_S)$, and we write $p = (\eta, |\cdot|_{\mathbf{u}, S})$ for some $\mathbf{u} = (u_1, \dots, u_r) \in \text{relint}(\Delta^{r-1})$ (cf. (2.6)). By Lemma 4.4, we get

$$g(p) = u_1 g(v_{j_1}) + \dots + u_r g(v_{j_r}) = \sum_{1 \leq b \leq r, b \neq a} u_b.$$

Since $\mathbf{u} \in \text{relint}(\Delta^{r-1})$, we have $u_1 > 0, \dots, u_r > 0$. Since $u_1 + \dots + u_r = 1$, it follows that $0 < \sum_{1 \leq b \leq r, b \neq a} u_b < 1$. Thus $0 < g(p) < 1$.

Next, we take any $q \in \Delta_T$. Let $v_{\ell_1}, \dots, v_{\ell_t}$ be the vertices of Δ_T . By the definition of Δ_T (cf. (2.6)), there exist $\mathbf{u}' = (u'_1, \dots, u'_t)$ with $u'_1 \geq 0, \dots, u'_t \geq 0$ and $u'_1 + \dots + u'_t = 1$ such that $q = (\eta, |\cdot|_{\mathbf{u}', T})$. Let \mathcal{U}' be a sufficiently small Zariski open neighborhood of the generic point of T . Recall that $f_{j_a} = \tilde{\eta}_{j_a}/\tilde{\eta}_0$ as a rational function on \mathcal{X} and that $\tilde{\eta}_0$ does not vanish at the generic point of any stratum. Thus $\text{div}(\tilde{\eta}_0)|_{\mathcal{U}'} = 0$. It follows that the horizontal part of $\text{div}(f_{j_a})|_{\mathcal{U}'}$ is effective. We apply Lemma 5.1, and then we get

$$g(q) \geq u'_1 g(v_{\ell_1}) + \dots + u'_t g(v_{\ell_t}).$$

Since $v_{\ell_1} \neq v_{j_a}, \dots, v_{\ell_t} \neq v_{j_a}$, Lemma 4.4 gives us $g(v_{\ell_1}) \geq 1, \dots, g(v_{\ell_t}) \geq 1$. It follows that $u'_1 g(v_{\ell_1}) + \dots + u'_t g(v_{\ell_t}) \geq 1$. Thus we have $g(q) \geq 1$.

Therefore, we have $g(\text{relint}(\Delta_S)) \subseteq (0, 1)$, and $g(\Delta_T) \subset [1, +\infty)$. Thus, in particular, $g(\text{relint}(\Delta_S)) \cap g(\text{relint}(\Delta_T)) = \emptyset$. This completes the proof. \square

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